Escape rates for rotor walks in $\mathbb{Z}^d$

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Abstract

Rotor walk is a deterministic analogue of random walk. We study its recurrence and transience properties on $\mathbb{Z}^d$ for the initial configuration of all rotors aligned. If n particles in turn perform rotor walks starting from the origin, we show that the number that escape (i.e., never return to the origin) is of order n in dimensions $d \geq 3$, and of order $n/\log n$ in dimension 2.

1 Introduction

In a *rotor walk* on a graph, the successive exits from each vertex follow a prescribed periodic sequence. For instance, in the square grid $\mathbb{Z}^2$, successive exits could repeatedly cycle through the sequence North, East, South West. Such walks were first studied in [9, 10] as a model of mobile agents exploring a territory, and in [6] as a model of self-organized criticality. In a lecture at Microsoft in 2003 [7], Jim Propp proposed rotor walk as a deterministic analogue of random walk, which naturally invited the question of whether rotor walk is recurrent in dimension 2 and transient in dimensions 3 and higher. One direction was settled immediately by Oded Schramm, who showed that rotor walk is "at least as recurrent" as random walk. Schramm's elegant argument, which we recall below, applies to any initial rotor configuration ρ.

The other direction is more subtle because it depends on ρ. Angel and Holroyd [1] showed that for all d there exist initial rotor configurations on $\mathbb{Z}^d$ such that rotor walk is recurrent. These special configurations are primed to send particles initially back toward the origin. The purpose of this note is to analyze the case $\rho = \uparrow$ when all rotors send their first particle in the same direction. To measure how transient this configuration is, we run n rotor walks starting from the origin and record whether each returns to the origin or escapes to infinity. We show that the number of escapes is of order n in dimensions $d \geq 3$, and of order $n/\log n$ in dimension 2.

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Write $\mathcal{E} = \{\pm e_1, \dots, \pm e_d\}$ for the set of $2d$ cardinal directions in $\mathbb{Z}^d$, and let $\mathcal{C}$ be the set of cyclic permutations of $\mathcal{E}$. A *rotor mechanism* is a map $m : \mathbb{Z}^d \to \mathcal{C}$, and a *rotor configuration* is a map $\rho : \mathbb{Z}^d \to \mathcal{E}$. A *rotor walk* is a sequence of vertices $x_0, x_1, \dots \in \mathbb{Z}^d$ and rotor configurations $\rho_0, \rho_1, \dots$ such that for all $n \geq 0$

$$x_{n+1} = x_n + \rho_n(x_n).$$

and

$$\rho_{n+1}(x_n) = m(x_n)(\rho_n(x_n))$$

and $\rho_{n+1}(x) = \rho_n(x)$ for all $x \neq x_n$.

For example in $\mathbb{Z}^2$, each rotor $\rho(x)$ points North, South, East or West. An example of a rotor mechanism is the permutation North $\mapsto$ East $\mapsto$ South $\mapsto$ West $\mapsto$ North at all $x \in \mathbb{Z}^2$. The resulting rotor walk in $\mathbb{Z}^2$ has the following description: A particle repeatedly steps in the direction indicated by the rotor at its current location, and then this rotor turns 90 degrees clockwise. Note that this "prospective" convention — move the particle before updating the rotor — differs from the "retrospective" convention of past works such as [2, 1]. In the prospective convention, $\rho(x)$ indicates where the next particle will step from x, instead of where the previous particle stepped. The prospective convention is often more convenient when studying questions of recurrence and transience.

In this paper we fix once and for all a rotor mechanism m on $\mathbb{Z}^d$. Now depending on the rotor configuration ρ, one of two things can happen to a rotor walk started from the origin:

1. The walk eventually returns to the origin; or

2. The walk never returns to the origin, and visits each vertex in $\mathbb{Z}^d$ only finitely often.

Indeed, if any site were visited infinitely often, then each of its neighbors must be visited infinitely often, and so the origin itself would be visited infinitely often. In case 2 we say that the walk "escapes to infinity." Note that after the walk has either returned to the origin or escaped to infinity, the rotors are in a new configuration.

We say that ρ is *recurrent* if the rotor walk with initial configuration ρ returns to the origin infinitely often ($x_n = o$ for infinitely many n); otherwise, we say that ρ is *transient*. To quantify the degree of transience, consider the following experiment: let each of n particles in turn perform rotor walk starting from the origin until either returning to the origin or escaping to infinity. Denote by $I(\rho, n)$ the number of walks that escape to infinity. (Importantly, we do not reset the rotors in between trials!)

Schramm [8] proved that for any ρ,

$$\limsup_{n \to \infty} \frac{I(\rho, n)}{n} \leq \alpha_d \tag{1}$$

where α_d is the probability that simple random walk in $\mathbb{Z}^d$ does not return to the origin. Our first result gives a corresponding lower bound for the initial configuration $\uparrow$ in which all rotors start pointing in the same direction: $\uparrow(x) = e_d$ for all $x \in \mathbb{Z}^d$.

Theorem 1. *For the rotor walk on $\mathbb{Z}^d$ with $d \geq 3$ with all rotors initially aligned $\uparrow$, a positive fraction of particles escape to infinity; that is,*

$$\liminf_{n \to \infty} \frac{I(\uparrow, n)}{n} > 0.$$

One cannot hope for such a result to hold for an arbitrary ρ: Angel and Holroyd [1] discovered rotor configurations ρ_{rec} in all dimensions such that $I(\rho_{\mathrm{rec}}, n) = 0$ for all n.

Our next result concerns the fraction of particles that escape in dimension 2: for any rotor configuration ρ this fraction is at most $\frac{\pi/2}{\log n}$, and for the initial configuration $\uparrow$ it is at least $\frac{c}{\log n}$ for some $c > 0$.

Theorem 2. *For rotor walk in $\mathbb{Z}^2$ with any rotor configuration ρ, we have*

$$\limsup_{n\to\infty} \frac{I(\rho, n)}{n/\log n} \leq \frac{\pi}{2}.$$

Moreover, if all rotors are initially aligned $\uparrow$, then

$$\liminf_{n\to\infty} \frac{I(\uparrow, n)}{n/\log n} > 0.$$

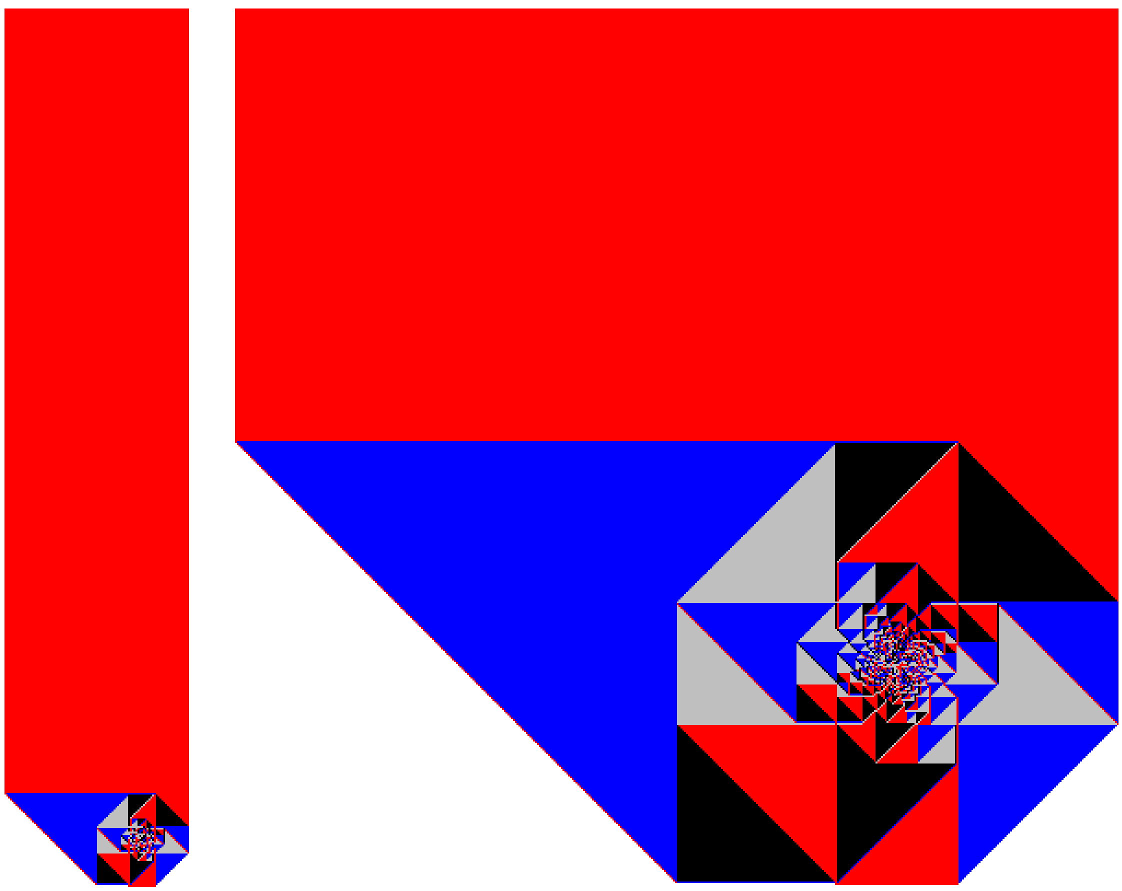

Figure 1: The configuration of rotors in $\mathbb{Z}^2$ after n particles started at the origin have escaped to infinity, with initial configuration $\uparrow$ (that is, all rotors send their first particle North). Top: $n = 100$; Bottom: $n = 480$. Each non-white pixel represents a point in $\mathbb{Z}^2$ that was visited at least once, and its color indicates the direction of its rotor.

2 Schramm's argument

One way to estimate the number of escapes to infinity of a rotor walk is to look at how many particles exit a large ball before returning to the origin. Let

$$\mathcal{B}_r = \{x \in \mathbb{Z}^d : ||x||_2 < r\}$$

be the set of lattice points in the open ball of radius r centered at the origin. Consider rotor walk started from the origin and stopped on hitting the boundary

$$\partial\mathcal{B}_r = \{y \in \mathbb{Z}^d \, : \, y \notin \mathcal{B}_r \text{ and } y \sim x \text{ for some } x \in \mathcal{B}_r\}.$$

Since $\mathcal{B}_r$ is a finite connected graph, this walk stops in finitely many steps.

Starting from initial rotor configuration ρ, let each of n particles in turn perform rotor walk starting from the origin until either returning to the origin or exiting the ball $\mathcal{B}_r$. Denote by $I_r(\rho, n)$ the number of particles that exit $\mathcal{B}_r$. The next lemmas give convergence and monotonicity of this quantity.

Lemma 3. *[3, Lemma 18] For any rotor configuration ρ and any $n \in \mathbb{N}$, we have $I_r(\rho, n) \to I(\rho, n)$ as $r \to \infty$.*

Proof. Let r_1 be such that all particles which return to the origin do not leave $\mathcal{B}_{r_1}$, and let r_2 be such that all particles which escape to infinity never return to $\mathcal{B}_{r_1}$ after they leave $\mathcal{B}_{r_2}$. If $r \geq r_2$, then stopping particles when they hit $\partial\mathcal{B}_r$ has no effect on the rotors inside $\mathcal{B}_{r_1}$, and therefore has no effect on whether future particles return to the origin. Hence $I_r(\rho, n) = I(\rho, n)$ for all $r \geq r_2$. □

For the next lemma we recall the *abelian property* of rotor walk [2, Lemma 3.9]. Let A be a finite subset of $\mathbb{Z}^d$. In an experiment of the form "run n rotor walks from prescribed starting points until they exit A," suppose that we repeatedly choose a particle in A and ask it to take a rotor walk step. Regardless of our choices, all particles will exit A in finitely many steps; for each $x \in A^c$, the number of particles that stop at x does not depend on the choices; and for each $x \in A$, the number of times we pointed to a particle at x does not depend on the choices.

Lemma 4. *[3, Lemma 19] For any rotor configuration ρ, any $n \in \mathbb{N}$ and any $r < R$, we have $I_R(\rho, n) \leq I_r(\rho, n)$.*

Proof. By the abelian property, we may compute $I_R(\rho, n)$ in two stages. First stop particles when they reach $\partial\mathcal{B}_r \cup \{o\}$, and then let the $I_r(\rho, n)$ particles stopped on $\partial\mathcal{B}_r$ continue walking until they reach $\partial\mathcal{B}_R \cup \{o\}$. Therefore at most $I_r(\rho, n)$ particles stop in $\partial\mathcal{B}_R$. □

Oded Schramm's upper bound (1) begins with the observation that if $2dm$ particles at a single site $x \in \mathbb{Z}^d$ each take a single rotor walk step, the result will be that m particles move to each of the $2d$ neighbors of x. Fix $r, m \in \mathbb{N}$ and consider $N = (2d)^r m$ particles at the origin. Let each particle take a single rotor walk step. Then repeat $r - 1$ times the following operation: let each particle that is not at the origin take a single rotor walk step. The result is that for each path of length $\ell \leq r$ starting and ending at the origin, exactly $(2d)^{-\ell} N$ particles traverse this path. Therefore the number of particles now at the origin is

$$N \sum_{\gamma : o \to o, \, |\gamma| \leq r} (2d)^{-|\gamma|} = Np$$

where $p = \mathbb{P}(T_o^+ \leq r)$ is the probability that simple random walk returns to the origin by time r. Now letting each particle that is not at the origin continue performing rotor walk until hitting $\partial\mathcal{B}_r \cup \{o\}$, the number of particles that stop in $\partial\mathcal{B}_r$ is at most $N(1-p)$, so

$$\frac{I_r(\rho, N)}{N} \leq 1 - p.$$

This holds for every N which is an integer multiple of $(2d)^r$. For general n, let N be the smallest multiple of $(2d)^r$ that is $\geq n$. Then

$$\frac{I_r(\rho, n)}{n} \leq \frac{I_r(\rho, N)}{N - (2d)^r}$$

The right side is at most $(1-p)(1 + 2(2d)^r/N)$, so

$$\limsup_{n\to\infty} \frac{I(\rho, n)}{n} \leq \limsup_{n\to\infty} \frac{I_r(\rho, n)}{n} \leq 1 - p = \mathbb{P}(T_o^+ > r).$$

As $r \to \infty$ the right side converges to α_d, completing the proof of (1).

See Holroyd and Propp [3, Theorem 10] for an extension of Schramm's argument to a general irreducible Markov chain with rational transition probabilities.

3 An odometer estimate for balls in all dimensions

To estimate $I_r(\rho, n)$, consider now a slightly different experiment. Let each of n particles started at the origin perform rotor walk until hitting $\partial\mathcal{B}_r$. (The difference is that we do not stop the particles on returning to the origin!) Define the *odometer function* u_n^r by

$$u_n^r(x) = \text{total number of exits from } x \text{ by } n \text{ rotor walks stopped on hitting } \partial\mathcal{B}_r.$$

Note that $u_n^r(x)$ counts the total number of exits (as opposed to the net number).

Now we relate the two experiments.

Lemma 5. *For any $r > 0$ and $n \in \mathbb{N}$ and any initial rotor configuration ρ, we have*

$$I_r(\rho, u_n^r(o)) = n.$$

Proof. Starting with $N = u_n^r(o)$ particles at the origin, consider the following two experiments:

1. Let n of the particles in turn perform rotor walk until hitting $\partial\mathcal{B}_r$.

2. Let N of the particles in turn perform rotor walk until hitting $\partial\mathcal{B}_r \cup \{o\}$.

By the definition of u_n^r, in the first experiment the total number of exits from the origin is exactly N. Therefore the two experiments have exactly the same outcome: n particles reach $\partial\mathcal{B}_r$ and $N - n$ remain at the origin. $\square$

Our next task is to estimate u_n^r. We begin by introducing some notation. Given a function f on $\mathbb{Z}^d$, its *gradient* is the function on directed edges given by

$$\nabla f(x, y) := f(y) - f(x).$$

Given a function κ on directed edges of $\mathbb{Z}^d$, its *divergence* is the function on vertices given by

$$\operatorname{div} \kappa(x) := \frac{1}{2d} \sum_{y \sim x} \kappa(x, y)$$

where the sum is over the $2d$ nearest neighbors of x. The *discrete Laplacian* of f is the function

$$\Delta f(x) := \operatorname{div} (\nabla f)(x) = \frac{1}{2d} \sum_{y \sim x} f(y) - f(x).$$

We recall some results from [5].

Lemma 6. *[5, Lemma 5.1] For a directed edge (x, y) in $\mathbb{Z}^d$, denote by $\kappa(x, y)$ the net number of crossings from x to y by n rotor walks started at the origin and stopped on exiting $\mathcal{B}_r$. Then*

$$\nabla u_n^r(x, y) = -2d\, \kappa(x, y) + R(x, y)$$

for some edge function R satisfying $|R(x, y)| \leq 4d - 2$ for all edges (x, y).

Denote by $(X_j)_{j \geq 0}$ the simple random walk in $\mathbb{Z}^d$, whose increments are independent and uniformly distributed on $\mathcal{E} = \{\pm e_1, \ldots, \pm e_d\}$. Let $T = \min\{j \,:\, X_j \notin \mathcal{B}_r\}$ be the first exit time from the ball of radius r. For $x, y \in \mathcal{B}_r$, let

$$G_r(x, y) = \mathbb{E}_x \#\{j < T | X_j = y\}$$

be the expected number of visits to y by a simple random walk started at x before time T. The following well known estimates can be found in [4, Prop. 1.5.9, Prop. 1.6.7]: for a constant a_d depending only on d,

$$G_r(x, o) = \begin{cases} a_d(|x|^{2-d} - r^{2-d}) + O(|x|^{1-d}), & d \geq 3 \\ \frac{2}{\pi}(\log r - \log |x|) + O(|x|^{-1}), & d = 2. \end{cases} \tag{2}$$

We will also use [4, Theorem 1.6.6] the fact that in dimension 2,

$$G_r(o, o) = \frac{2}{\pi} \log r + O(1). \tag{3}$$

(As usual, we write $f(n) = \Theta(g(n))$ (respectively, $f(n) = O(g(n))$) to mean that there is a constant $0 < C < \infty$ such that $1/C < f(n)/g(n) < C$ (respectively, $f(n)/g(n) < C$) for all sufficiently large n. Here and in what follows, the constants implied in $O()$ and $\Theta()$ notation depend only on the dimension d.)

The next lemma bounds the L^1 norm of the discrete gradient of the function $G_r(x, \cdot)$.

Lemma 7. *[5, Lemma 5.6] Let $x \in \mathcal{B}_r$ and let $\rho = r + 1 - |x|$. Then for some C depending only on d,*

$$\sum_{y \in \mathcal{B}_r} \sum_{z \sim y} |G_r(x, y) - G_r(x, z)| \leq C \rho \log \frac{r}{\rho}.$$

The next lemma is proved in the same way as the inner estimate of [5, Theorem 1.1]. Let $f(x) = n G_r(x, o)$.

Lemma 8. *In $\mathbb{Z}^d$, let $x \in \mathcal{B}_r$ and $\rho = r + 1 - |x|$. Then,*

$$|u_n^r(x) - f(x)| \leq C\rho \log \frac{r}{\rho} + 8d^2.$$

where u_n^r is the odometer function for n particles performing rotor walk stopped on exiting $\mathcal{B}_r$, and C is the constant in Lemma 7.

Proof. If we consider the rotor walk stopped on exiting $\mathcal{B}_r$, all sites that have positive odometer value have been hit by particles. Using notation of Lemma 6, we notice that since the net number of particles to enter a site $x \neq o$ not on the boundary is zero, we have $2d \operatorname{div} \kappa(x) = 0$. For the origin, $2d \operatorname{div} \kappa(o) = n$. Also, the odometer function vanishes on the boundary, since the boundary does not emit any particles.

Write $u = u_n^r$. Using the definition of κ in Lemma 6, we see that

$$\begin{aligned} \Delta u(x) &= \operatorname{div} R(x), \;\; x \neq o, && (4) \\ \Delta u(o) &= -n + \operatorname{div} R(o). && (5) \end{aligned}$$

Then $\Delta f(x) = 0$ for $x \in \mathcal{B}_r \setminus \{o\}$ and $\Delta f(o) = -n$ and f vanishes on $\partial \mathcal{B}_r$.

Since $u(X_T)$ is equal to 0, we have

$$u(x) = \sum_{k \geq 0} \mathbb{E}_x (u(X_{k \wedge T}) - u(X_{(k+1) \wedge T})).$$

Also, since the k^{th} term in the sum is zero when $T \leq k$

$$\mathbb{E}_x (u(X_{k \wedge T}) - u(X_{(k+1) \wedge T}) | \mathcal{F}_{k \wedge T}) = -\Delta u(X_k) 1_{\{T > k\}}$$

where $\mathcal{F}_j = \sigma(X_0, \ldots, X_j)$ is the standard filtration for the random walk.

Taking expectation of the conditional expectations and using (4) and (5), we get

$$\begin{aligned} u(x) &= \sum_{k \geq 0} \mathbb{E}_x \left[1_{\{T > k\}} (n 1_{\{X_k = o\}} - \operatorname{div} R(X_k)) \right] \\ &= n\, \mathbb{E}_x \#\{k < T | X_k = 0\} - \sum_{k \geq 0} \mathbb{E}_x \left[1_{\{T > k\}} \operatorname{div} R(X_k) \right]. \end{aligned}$$

So,

$$u(x) - f(x) = -\frac{1}{2d} \sum_{k \geq 0} \mathbb{E}_x \left[1_{\{T > k\}} \sum_{z \sim X_k} R(X_k, z) \right].$$

Since the random walk exits $\mathcal{B}_r$ with probability at least $\frac{1}{2d}$ every time it reaches a site adjacent to the boundary $\partial \mathcal{B}_r$, the expected time spent adjacent to the boundary before time T is at most $2d$. Furthermore, since $|R| \leq 4d$, the terms with $z \in \partial \mathcal{B}_r$ contribute at most $16d^3$ to the sum. Thus,

$$|u(x) - f(x)| \leq \frac{1}{2d} \left| \sum_{k \geq 0} \mathbb{E}_x \left[\sum_{\substack{y, z \in B_r \\ y \sim z}} 1_{\{T > k\} \cap \{X_k = y\}} R(y, z) \right] \right| + 8d^2. \tag{6}$$

Note that for $y \in \mathcal{B}_r$ we have $\{X_k = y\} \cap \{T > k\} = \{X_{k\wedge T} = y\}$. Considering $p_k(y) = \mathbb{P}_x(X_{k\wedge T} = y)$, and noting that R is antisymmetric (because of antisymmetry in Lemma 6), we see that

$$\begin{aligned}\sum_{\substack{y,z\in B_r\\ y\sim z}} p_k(y)R(y,z) &= -\sum_{\substack{y,z\in B_r\\ y\sim z}} p_k(y)R(y,z)\\ &= \sum_{\substack{y,z\in B_r\\ y\sim z}} \frac{p_k(y)-p_k(z)}{2}R(y,z).\end{aligned}$$

Summing over k in (6) and using the fact that $|R| \le 4d$, we conclude that

$$|u(x) - f(x)| \le \sum_{\substack{y,z\in B_r\\ y\sim z}} |G(x,y) - G(x,z)| + 8d^2.$$

The result now follows from the estimate of the gradient of Green's function in Lemma 7. $\square$

Now we make our choice of radius, $r = n^{1/(d-1)}$. The next lemma shows that for this value of r, the support of the odometer function contains a large sphere.

Lemma 9. *There exists a constant $\beta > 0$ depending only on d, such that for any initial rotor configuration and $r = n^{1/(d-1)}$ we have $u_n^r(x) > 0$ for all $x \in \partial\mathcal{B}_{\beta r}$.*

Proof. For $x \in \partial\mathcal{B}_{\beta r}$ we have $\beta r \le |x| \le \beta r + 1$. By Lemma 8 we have

$$|u_n^r(x) - f(x)| \le C(1-\beta) r \log\frac{1}{1-\beta}.$$

for a constant C depending only on d. To lower bound $f(x)$ we use (2): in dimensions $d \ge 3$, we have

$$f(x) = nG_r(x,o) \sim n a_d(|x|^{2-d} - r^{2-d}) \sim a_d(\beta^{2-d} - 1) n r^{2-d}.$$

Since $r = nr^{2-d}$, we can take $\beta > 0$ sufficiently small so that

$$a_d(\beta^{2-d} - 1) n r^{(2-d)} > 2C(1-\beta) r \log\frac{1}{1-\beta}$$

and hence $u_n^r(x) > 0$.

In dimension 2, we have $r = n$ and $nG_n(x,o) = n\frac{2}{\pi}\ln\frac{1}{\beta} + O(1)$, by (2). So for β small enough, independent of n, we have that

$$nG_n(x,o) = n\frac{2}{\pi}\log\frac{1}{\beta} + O(1) > C(1-\beta) n \log\frac{1}{1-\beta},$$

which implies that $u_n^n(x) > 0$. $\square$

Identify $\mathbb{Z}^d$ with $\mathbb{Z}^{d-1} \times \mathbb{Z}$ and call each set of the form $(x_1, \ldots, x_{d-1}) \times \mathbb{Z}$ a "column." Starting n particles at the origin and letting them each perform rotor walk until exiting $\mathcal{B}_r$ where $r = n^{1/(d-1)}$, let $\mathrm{col}(\rho, n)$ be the number of distinct columns that are visited. That is,

$$\mathrm{col}(\rho,n) = \#\{(x_1,\ldots,x_{d-1}) : u_n^r(x_1, x_2, \ldots, x_d) > 0 \text{ for some } x_d \in \mathbb{Z}\}.$$

By Lemma 9, every site of $\partial\mathcal{B}_{\beta r}$ is visited at least once, so

$$\begin{aligned} \mathrm{col}(\rho, n) &\geq \#\{(x_1, \ldots, x_{d-1}) : (x_1, x_2, \ldots, x_d) \in \partial\mathcal{B}_{\beta r} \text{ for some } x_d \in \mathbb{Z}\} \\ &\geq C(\beta r)^{d-1} = \Theta(n). \end{aligned} \tag{7}$$

All results so far have not made any assumptions on the initial configuration. The next lemma assumes the initial rotor configuration to be $\uparrow$. The important property of this initial condition for us is that the first particle to visit a given column travels straight along that column in direction e_d thereafter.

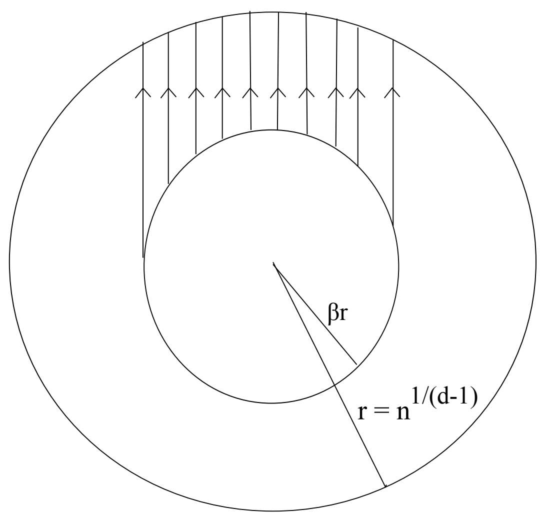

Figure 2: Diagram for the proof of Lemma 10. The first visit to each column results in an escape along that column, so at least $\mathrm{col}(\uparrow, n)$ particles escape.

Lemma 10. *In $\mathbb{Z}^d$ with initial rotor configuration $\uparrow$, we have*

$$I_R(\uparrow, u_n^r(o)) \geq \mathrm{col}(\uparrow, n)$$

for all $R \geq r$.

Proof. By the abelian property of rotor walk, we may compute $I_R(\rho, u_n^r(o))$ in two stages. First we stop the particles when they first hit $\partial\mathcal{B}_r \cup \{o\}$. Then we let all the particles stopped on $\partial\mathcal{B}_r$ continue walking until they hit $\partial\mathcal{B}_R \cup \{o\}$. By Lemma 5, exactly n particles stop on $\partial\mathcal{B}_r$ during the first stage, and therefore $\mathrm{col}(\uparrow, n)$ distinct columns are visited during the first stage. Because the initial rotors are $\uparrow$, the first particle to visit a given column travels straight along that column to hit $\partial\mathcal{B}_R$ (Figure 2). Therefore the number of particles stopping in $\partial\mathcal{B}_R$ is at least $\mathrm{col}(\uparrow, n)$. □

4 The transient case: Proof of Theorem 1

In this section we consider $\mathbb{Z}^d$ for $d \geq 3$. We will prove Theorem 1 by comparing the number of escapes $I(\uparrow, n)$ with $\mathrm{col}(\uparrow, n)$.

Let $r = n^{1/(d-1)}$ and $N = u_n^r(o)$. By the transience of simple random walk in $\mathbb{Z}^d$ for $d \geq 3$ we have

$$f(o) = nG_r(o, o) = \Theta(n).$$

By Lemma 8 we have $|N - f(o)| = O(r)$ and hence $N = \Theta(n)$. By Lemmas 3 and 10 we have $I(\uparrow, N) \geq \mathrm{col}(\uparrow, n)$. Recalling (7) that $\mathrm{col}(\uparrow, n) = \Theta(n)$ and that $I(\uparrow, n)$ is nondecreasing in n, we conclude that there is a constant $c > 0$ depending only on d such that for all sufficiently large n

$$\frac{I(\uparrow, n)}{n} > c$$

which completes the proof.

5 The recurrent case: Proof of Theorem 2

In this section we work in $\mathbb{Z}^2$ and take $r = n$. We start by estimating the odometer function at the origin for the rotor walk stopped on exiting $\mathcal{B}_n$.

Lemma 11. *For any initial rotor configuration in $\mathbb{Z}^2$ we have*

$$u_n^n(o) = \frac{2}{\pi} n \log n + O(n).$$

Proof. By (3), we have $f(o) = nG_n(o, o) = n(\frac{2}{\pi}\log n + O(1))$, and $|u_n^n(o) - f(o)| = O(n)$ by Lemma 8. $\square$

Turning to the proof of the upper bound in Theorem 2, let $N = u_n^n(o)$. By Lemmas 3 and 4, $I(\rho, N) \leq I_n(\rho, N)$. By Lemma 5, $I_n(\rho, N) = n$. Now by Lemma 11, $\frac{N}{\log N} = \frac{(2/\pi)n\log n + O(n)}{\log n + O(\log\log n)} = (\frac{2}{\pi} + o(1))n$, hence

$$\frac{I(\rho, N)}{N/\log N} \leq \frac{n}{(\frac{2}{\pi} + o(1))n} = \frac{\pi}{2} + o(1).$$

Since $I(\rho, n)$ is nondecreasing in n, the desired upper bound follows.

To show the lower bound for $\uparrow$ we use lemmas 3 and 10 along with (7)

$$I(\uparrow, N) = \lim_{R\to\infty} I_R(\uparrow, N) \geq \mathrm{col}(\uparrow, n) \geq \beta n = \Theta(\frac{N}{\log N}).$$

Since $I(\rho, n)$ is nondecreasing in n the desired lower bound follows.

Remark. The proofs of the lower bounds in Theorems 1 and 2 apply to a slightly more general class of rotor configurations than $\uparrow$. Given a rotor configuration ρ, the *forward path* from x is the path $x = x_0, x_1, x_2, \ldots$ defined by $x_{k+1} = x_k + \rho(x_k)$ for $k \geq 0$. Let us say that $x \in \partial\mathcal{B}_r$ has a *simple path to infinity* if the forward path from x is simple (that is, all x_k are distinct) and $x_k \notin \partial\mathcal{B}_r$ for all $k \geq 1$. The proofs we have given for $\uparrow$ remain valid for ρ as long as there is a constant C and a sequence of radii $r_1, r_2, \ldots$ with $r_{i+1}/r_i < C$, such that for each i, at least r_i^{d-1}/C sites on $\partial\mathcal{B}_{r_i}$ have disjoint simple paths to infinity. For instance, the rotor configuration

$$\rho(x) = \begin{cases} \alpha, & x_d \geq 0 \\ \beta, & x_d < 0 \end{cases}$$

satisfies this condition as long as $(\alpha, \beta) \neq (-e_d, +e_d)$.

6 Some open questions

We conclude with a few natural questions.

- When is Schramm's bound attained? In $\mathbb{Z}^d$ for $d \geq 3$ with rotors initially aligned in one direction, is the escape rate for rotor walk asymptotically equal to the escape probability of the simple random walk? Theorem 1 shows that the escape rate is positive.

- Does every transient graph have a rotor configuration ρ for which a positive fraction of particles escape to infinity, that is, $\liminf_{n\to\infty} \frac{I(\rho,n)}{n} > 0$?

- Let us choose initial rotors $\rho(x)$ for $x \in \mathbb{Z}^d$ independently and uniformly at random from $\mathcal{E} = \{\pm e_1, \ldots, \pm e_d\}$. Is the resulting rotor walk recurrent in dimension 2 and transient in dimensions $d \geq 3$? Angel and Holroyd [1, Corollary 6] prove that two initial configurations differing in only a finite number of rotors are either both recurrent or both transient. Hence the set of recurrent ρ is a tail event and consequently has probability 0 or 1.

- Starting from initial rotor configuration $\uparrow$ in $\mathbb{Z}^2$, let ρ_n be the rotor configuration after n particles have escaped to infinity. Does $\rho_n(nx, ny)$ have a limit as $n \to \infty$? Figure 1 suggests that the answer is yes.

- Consider rotor walk in $\mathbb{Z}^2$ with a drift to the north: each rotor mechanism is period 5 with successive exits cycling through North, North, East, South, West. Is this walk transient for all initial rotor configurations?

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